MATH 2010B Advanced Calculus I Lecture Notes Week 11

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Last time... Directional Derivative

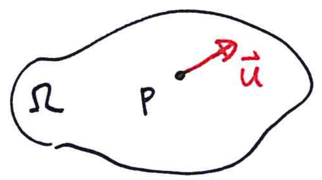
$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$

$$D_{\vec{u}} f(p) = \nabla f(p) \cdot \vec{u}$$

if f is diff. at p.

$\|\vec{u}\| = 1$

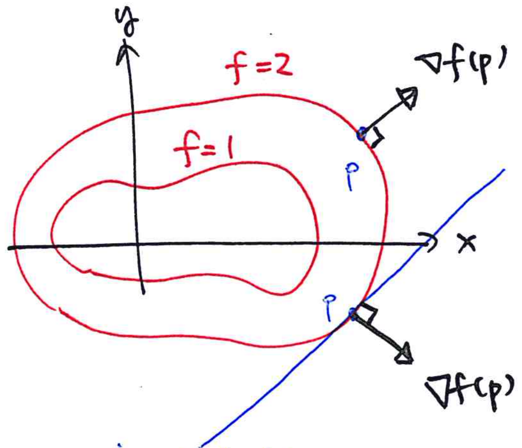
unit vector



Gradient ∇f and level sets

Thm: $\nabla f \perp$ to level sets of f (in all dimensions)

Level Curves in $\mathbb{R}^2$: $(n=2)$ $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$.

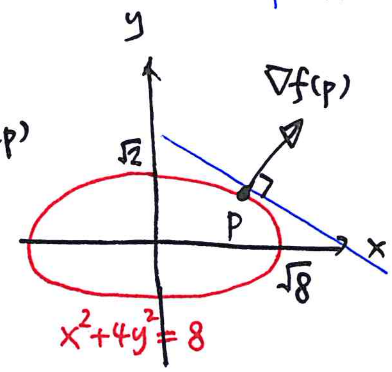


L_p : tangent line to $\{f=2\}$ at p

Example: Find the tangent line to

$$x^2 + 4y^2 = 8 \quad \text{at} \quad P = (2,1)$$

(ellipse).



Sol: The ellipse is the level set of

$$\{ f(x,y) = x^2 + 4y^2 = 8 \}$$

$$\nabla f(p) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \Big|_P$$

$$= (2x, 8y) \Big|_{P=(2,1)}$$

$$= (4, 8).$$

So, the equation for the tangent line at P:

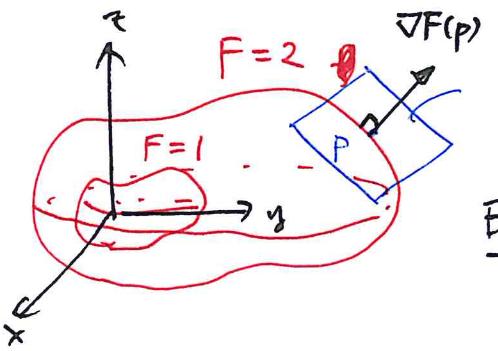
$$(4,8) \cdot (x,y) = (4,8) \cdot (2,1)$$

$$\Rightarrow \quad 4x + 8y = 8 + 8 = 16$$

$$\Rightarrow \quad \boxed{x + 2y = 4}$$

※

Level surface in $\mathbb{R}^3$: $(n=3)$ $F(x,y,z): \mathbb{R}^3 \to \mathbb{R}$.



tangent plane to level surface $\{F=2\}$ at P.

Example: Find an equation of the tangent plane to the surface

$$S = \{\cos \pi x - x^2 y + e^{xz} + yz = 4\}$$

at $P = (0,1,2)$. Check P lies on the surface.

Sol: Define $F(x,y,z) = \cos \pi x - x^2 y + e^{xz} + yz - 4$.

and thus $S = \{F = 0\}$.

$$\nabla F(P) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)\Big|_P$$

$$= \left(-\pi \sin \pi x - 2xy + z e^{xz},\ -x^2 + z,\ x e^{xz} + y \right)\Big|_P$$

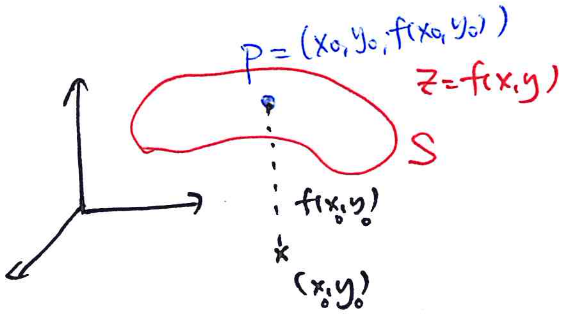
$$= (2,2,1).$$ normal to tangent plane

The equation for the tangent plane at P:

$$(2,2,1)\cdot(x,y,z) = (2,2,1)\cdot(0,1,2)$$

$$\Rightarrow \boxed{2x + 2y + z = 4}$$ ※

Recall: Graph of $z = f(x,y)$



Equation of tangent plane at P:

$$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)} \quad (*)$$

Idea: View $S = \{z = f(x,y)\}$ graph of $f(x,y)$

$= \{ \underbrace{z - f(x,y)}_{F(x,y,z)} = 0 \}$ level set of $F(x,y,z)$

Goal: Find the tangent plane at $P = (x_0, y_0, f(x_0,y_0))$.

$$\nabla F(P) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P$$

$$= (-f_x, -f_y, 1)\Big|_P$$

$$= (-f_x(x_0,y_0), -f_y(x_0,y_0), 1)$$

Equation is:

$$(-f_x(x_0,y_0), -f_y(x_0,y_0), 1) \cdot (x,y,z)$$
$$= (-f_x(x_0,y_0), -f_y(x_0,y_0), 1) \cdot (x_0, y_0, f(x_0,y_0))$$

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0). \quad \text{— } (*)$$

Example: Find the equation of the tangent plane to the quadric surface

$$Ax^2 + By^2 + Cz^2 = D$$

at $P = (x_0, y_0, z_0)$.

level surface $\{F = D\}$

Sol: Define $F(x,y,z) = Ax^2 + By^2 + Cz^2$.

$$\nabla F(P) = (2Ax, 2By, 2Cz)\Big|_P$$

$$= (2Ax_0, 2By_0, 2Cz_0)$$

Equation: $(2Ax_0)x + (2By_0)y + (2Cz_0)z =$

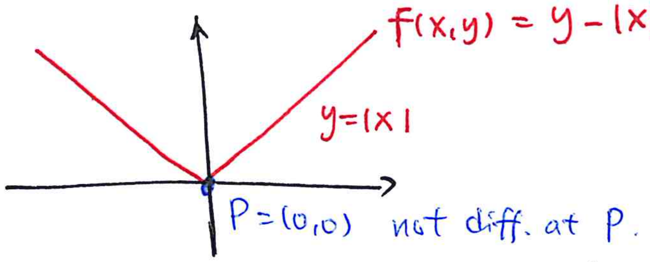
$$(2Ax_0)x_0 + (2By_0)y_0 + (2Cz_0)z_0 = 2D$$

P lies on the surface.

$$\boxed{(Ax_0)\,x + (By_0)\,y + (Cz_0)\,z = D.}$$ ✱

Q: When does this method fail?

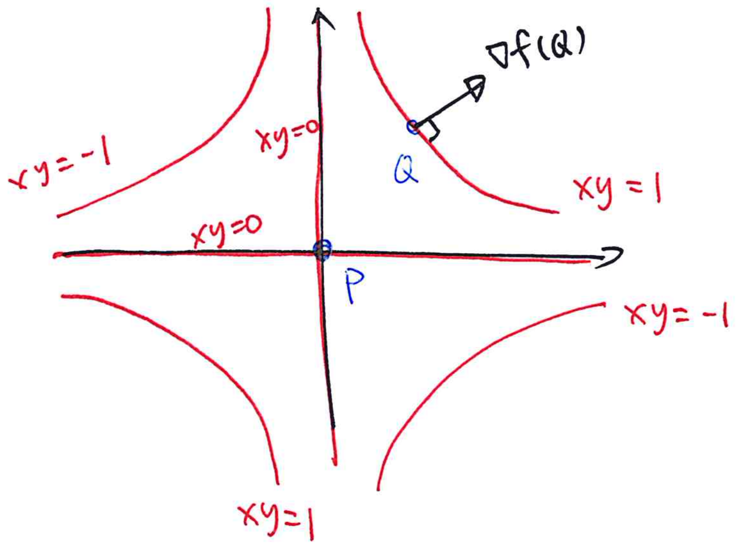
A: ① If $F(x,y,z)$ is not differentiable at P.



Note: In fact, no "tangent line" at P.

② If $\nabla F(P) = \vec{0}$, then you don't get a "normal vector".

2D case: $f(x,y) = xy : \mathbb{R}^2 \longrightarrow \mathbb{R}$. differentiable.



$\nabla f(x,y) = (y, x)$.

$\nabla f = \vec{0} \iff (x,y) = (0,0)$

At $P=(0,0)$, the level set $\{f=0\}$ is "singular".

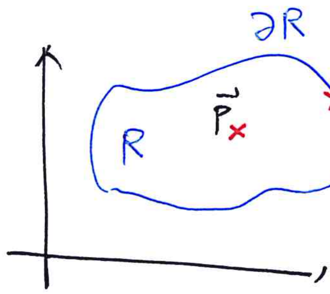
3D case: Ex: Understand ALL the level surfaces of $F(x,y,z) = x^2+y^2+z^2$.

Last time ... ∇f, related to tangent lines/planes

Optimization Revisited (2D)

Given $f = f(x,y)$, solve

$$\max/\min_{R} f(x,y)$$



If $\vec{p}$ is an interior extremum, then

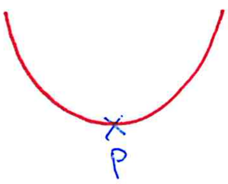
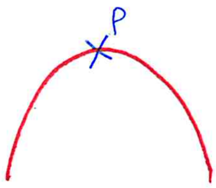
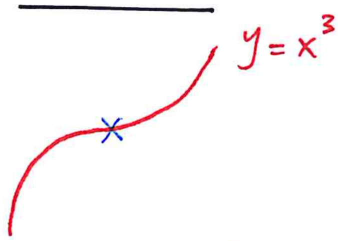
$$\boxed{\nabla f(\vec{p}) = \vec{0}}$$

1st order condition / 1st derivative test

$\vec{p}$ is called a critical point

Recall: (1D case) $f(x)$. p critical $\Rightarrow f'(p) = 0$.

(2nd derivative test)

$f''(p) > 0$	$f''(p) < 0$	$f''(p) = 0$
		
local min	local max	inflection pts. (degenerate case)

Q: Do you have a 2nd derivative test in 2D or higher dimensions?

1^{st} derivatives $\qquad$ 2^{nd} derivatives

$$
\begin{tikzcd}
 & & f_{xx} \\
 & f_x \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[r, "\frac{\partial}{\partial y}"'] & f_{xy} \\
f(x,y) \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & & \parallel \\
 & f_y \arrow[r, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & f_{yx} \\
 & & f_{yy}
\end{tikzcd}
$$

$f_{xy} = f_{yx}$ ← holds when $f \in C^2$

$\nabla f = (f_x, f_y)$

"gradient vector"

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

2x2 matrix symmetric

"Hessian matrix"

<u>Note</u>: So we need a notion for a 2x2 matrix A to be "$A > 0$" or "$A < 0$" ?

<u>Some linear algebra</u>

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad 2\times 2 \text{ symmetric matrix.}$$

<u>Assume</u>: A is non-singular, ie $\boxed{\det A \neq 0}$.

<u>Define</u>: Define a quadratic form Q associated to A

$$Q(\vec{x}) = \vec{x}^T A \vec{x}.$$

ie $$Q(x,y) = (x\ y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2$$

We say that

(1) A is <u>positive definite</u> (ie "$A > 0$") if $Q(\vec{x}) > 0$ if $\vec{x} \neq \vec{0}$.

(2) A is <u>negative definite</u> (ie "$A < 0$") if $Q(\vec{x}) < 0$ if $\vec{x} \neq \vec{0}$

(3) A is <u>indefinite</u> if it's not (1) or (2).

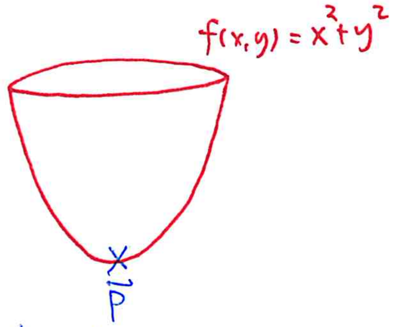
↑ new to 2D

<u>Special case</u>: $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ diagonal. $\left(\underset{\lambda_1 \lambda_2}{\overset{\parallel}{\det A}} \neq 0 \Rightarrow \lambda_1, \lambda_2 \neq 0 \right)$

(3 possibilities:)

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > 0$

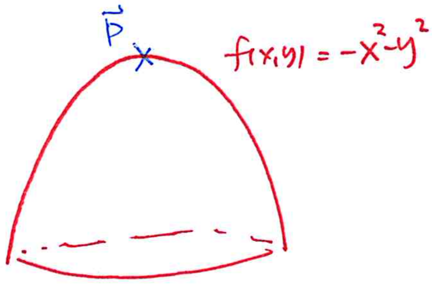
$Q(x,y) = x^2 + y^2 > 0$



local min.

$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} < 0$

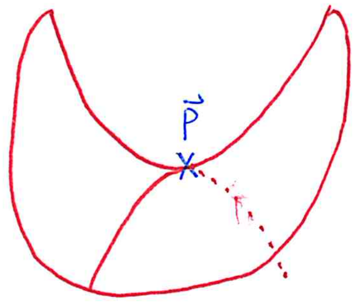
$Q(x,y) = -x^2 - y^2 < 0$



local max

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$Q(x,y) = x^2 - y^2$



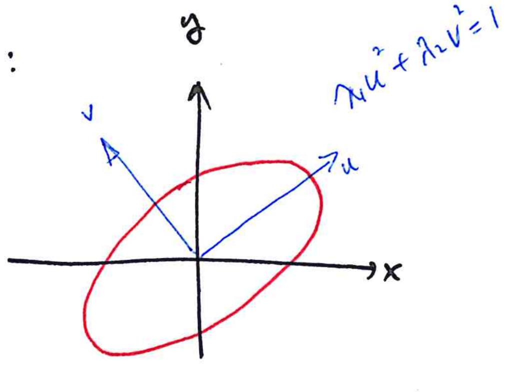
saddle point

Q: What if A is <u>NOT</u> diagonal?

good news: $\underset{\text{symmetric}}{A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}} \xrightarrow[\text{(change of coordinates)}]{\text{diagonalize}} D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$D = B^{-1} A B$. for some B.

<u>geometry</u>:



<u>Fact</u>: • $\det A = \det D = \lambda_1 \lambda_2$

$\begin{cases} \text{if } \lambda_1\lambda_2 > 0 \Rightarrow \lambda_1, \lambda_2 > 0 \text{ or } \lambda_1, \lambda_2 < 0 \\ \text{if } \lambda_1\lambda_2 < 0 \Rightarrow \lambda_1, \lambda_2 \text{ have different signs.} \end{cases}$

• $\underset{a+c}{\overset{\parallel}{\operatorname{tr} A}} = \operatorname{tr} D = \lambda_1 + \lambda_2$

Theorem: $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

(1) If $\det A = ac - b^2 > 0$, $a > 0$, then "$A > 0$"

(2) If $\det A = ac - b^2 > 0$, $a < 0$, then "$A < 0$".

(3) If $\det A = ac - b^2 < 0$, then A is indefinite.

Theorem (2nd Derivative Test)

Let $f(x,y)$ be a C^2 function.

Suppose $\vec{p}$ is a critical pt. of f, ie $\nabla f(\vec{p}) = \vec{0}$.

Let

$$Hf(\vec{p}) = \left.\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\right|_{\vec{p}}$$

2x2 symmetric matrix of numbers.

(Hessian at $\vec{p}$)

Then, we have the following classification:

(1) If $Hf(\vec{p}) > 0$ (ie. $f_{xx}f_{yy} - f_{xy}^2 > 0$, $f_{xx} > 0$) $\Rightarrow$ $\vec{p}$ is a local min.

(2) If $Hf(\vec{p}) < 0$ (ie. $f_{xx}f_{yy} - f_{xy}^2 > 0$, $f_{xx} < 0$) $\Rightarrow$ $\vec{p}$ is a local max

(3) If $Hf(\vec{p})$ is indefinite (ie $f_{xx}f_{yy} - f_{xy}^2 < 0$) $\Rightarrow$ $\vec{p}$ is a saddle pt.

assuming that $\vec{p}$ is a "non-degenerate" critical pt. ,ie $\det(Hf(\vec{p})) \neq 0$.

Example 1: Find and classify all the critical points of

$$f(x,y) = 2x^2 + y^2 + 4x - 4y + 5.$$

Sol: 1st order condition: $\boxed{\nabla f = 0}$

$$\begin{cases} 0 = f_x = 4x + 4 \\ 0 = f_y = 2y - 4 \end{cases} \Rightarrow \begin{matrix} x = -1 \\ y = 2 \end{matrix}$$

$\boxed{\vec{p} = (-1, 2) \text{ is the only critical pt.}}$

2^{nd} derivative test : " $Hf(\vec{p}) > 0$ or < 0 or indef. "

$$Hf(\vec{p}) = \left.\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\right|_{\vec{p}} = \left.\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\right|_{\vec{p}} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}.$$

diagonal with positive eigenvalues

$\Rightarrow$ $\vec{p}$ is a local min.

Example 2 : Do the same for

$$f(x,y) = xy\,e^{-x^2-y^2}.$$

Sol : Step 1 : Locate all critical pts.

$$\begin{cases} 0 = f_x = y e^{-x^2-y^2} + xy\cdot(-2x)e^{-x^2-y^2} \\ 0 = f_y = x e^{-x^2-y^2} + xy\cdot(-2y)e^{-x^2-y^2} \end{cases}$$

$$\Rightarrow \begin{cases} 0 = y - 2x^2y & \text{— (1)} \\ 0 = x - 2xy^2 & \text{— (2)} \end{cases}$$

If $y \neq 0$, (1) $\Rightarrow$ $2x^2 = 1$ $\Rightarrow$ $x = \pm\frac{1}{\sqrt{2}}$.

(2) $\Rightarrow$ $2y^2 = 1$ $\Rightarrow$ $y = \pm\frac{1}{\sqrt{2}}$.

4 critical pts: $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

If $y = 0$, (2) $\Rightarrow$ $x = 0$

1 critical pt : $(0, 0)$

5 critical pts.

Step 2: Classify them using 2^{nd} Derivative test.

2^{nd} derivatives: Recall: $\begin{cases} f_x = (y-2x^2y)e^{-x^2-y^2} \\ f_y = (x-2xy^2)e^{-x^2-y^2} \end{cases}$

$$f_{xx} = -4xe^{-x^2-y^2} + (y-2x^2y)(-2x)e^{-x^2-y^2}$$

$$= e^{-x^2-y^2}\left(-4x - 2xy(1-2x^2)\right)$$

$$f_{xy} = (1-2x^2)e^{-x^2-y^2} + (y-2x^2y)(-2y)e^{-x^2-y^2}$$

$$= e^{-x^2-y^2}\left((1-2y^2)(1-2x^2)\right)$$

$$f_{yx} = f_{xy}$$

$$f_{yy} = e^{-x^2-y^2}\left(-4y - 2xy(1-2y^2)\right)$$

At $(0,0)$,

$$Hf(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

det $= -1 < 0$

$\Rightarrow$ indefinite

$\Rightarrow$ $(0,0)$ is a saddle pt.

At $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$,

$$Hf\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = \begin{pmatrix} -2\sqrt{2}e^{-1} & 0 \\ 0 & -2\sqrt{2}e^{-1} \end{pmatrix}$$

$\Rightarrow$ negative def.

$\Rightarrow$ $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is local max

At $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $Hf > 0$ $\Rightarrow$ $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is local min.

At $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$$Hf\left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right) = \begin{pmatrix} -2\sqrt{2}e^{-1} & 0 \\ 0 & 2\sqrt{2}e^{-1} \end{pmatrix}$$

$\Rightarrow$ indefinite

$\Rightarrow$ $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is a saddle pt.

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a saddle pt.

Q: Why is 2^{nd} derivative test true?

1D case: Taylor approximation!

$f(x)$, at $x = 0$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 + \text{higher order terms}$$

$$\approx \underbrace{f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2}_{\text{quadratic approximation.}} \qquad \text{when } x \approx 0$$

$$f(x) \approx f(0) + \underbrace{\boxed{f'(0)\,x}}_{\substack{= \\ 0 \\ \text{since } f'(0)=0}} + \underbrace{\boxed{\frac{f''(0)}{2}x^2}}_{\substack{\geq \\ 0}} \geq f(0) \quad \Rightarrow 0 \text{ is a local min}$$

since $f''(0) > 0$

2D case: Taylor approximation!

$f(x,y)$, at $(x,y) = (0,0)$

$$f(x,y) = \underbrace{f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y + \frac{1}{2}\left(f_{xx}(0,0)\,x^2 + 2f_{xy}(0,0)\,xy + f_{yy}(0,0)\,y^2\right)}_{\text{quadratic approximations.}}$$

$+$ higher order terms.

$$\approx f(0,0) + \underbrace{\boxed{\nabla f(0,0)\cdot(x,y)}}_{\substack{= \\ 0 \\ \text{if } \nabla f(0,0) = (0,0)}} + \frac{1}{2}\underbrace{(x,y)\,Hf(0,0)\begin{pmatrix}x\\y\end{pmatrix}}_{\vec{x}^{T}A\vec{x} \geq 0} \geq f(0,0)$$

Ex: Try to prove Taylor's theorem (2D).